Problem Solving Strategies 4

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Q> We start with the positive integers $1, \dots, 4n-1$. In one move we will replace any two numbers by their difference. Prove that an even integer will be left after $4n-2$ steps.

Ans:- $2n-1$ even, $2n$ odd $\rightarrow$ Initially parity even

Two odd numbers a, b are chosen $\rightarrow |a-b|$ is even is added $\rightarrow$ even $\rightarrow +1$, odd $\rightarrow -2$

One odd and One even a, b are chosen $\rightarrow |a-b|$ is odd is added $\rightarrow$ even $\rightarrow -1$, odd $\rightarrow 0$

Two even number a, b are chosen $\rightarrow |a-b|$ is even is added $\rightarrow$ even $\rightarrow -1$, odd $\rightarrow 0$

odd or even $\leftarrow$ Parity of number of odd numbers is always even

So in the end it will also be even but after $4n-2$ steps we are left with one element. So number of odd numbers will be 0

So done

Starting from (x_0, y_0) with $0 < x_0 < y_0$

In each step we do, $x_{n+1} = \dfrac{x_n + y_n}{2}$, $y_{n+1} = \sqrt{x_{n+1} y_n}$

Assume $\rightarrow x_n < y_n$

$GM(a,b) = \sqrt{ab}$

$a < b \Rightarrow a^2 < ab \Rightarrow a < \sqrt{ab}$

$b^2 > ab \Rightarrow b > \sqrt{ab}$

$\Rightarrow x_{n+1} = \dfrac{x_n + y_n}{2}$ $\quad y_{n+1} = \sqrt{x_{n+1} y_n}$

$\rightarrow$ GM of x_{n+1} and y_n where $x_{n+1} < y_n$

$\Rightarrow x_{n+1} < y_n$

$\Rightarrow x_{n+1} < y_{n+1} < y_n$

$\Rightarrow x_{n+1} < y_{n+1}$

So $x_n < y_n \ \forall n$

Then, $y_{n+1} - x_{n+1} = \sqrt{x_{n+1} y_n} - \dfrac{x_n + y_n}{2} <$ in terms of y_n & x_n

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Q> We have a set $\{3, 4, 12\}$. In each step we choose two of the

Q) We have a set $\{3, 4, 12\}$. In each step we choose two of the numbers from the set a, b and replace them by, $0.6a - 0.8b$ and $0.8a + 0.6b$.
Can we reach $\{4, 6, 12\}$ from these steps?
Can we reach $\{x, y, z\}$ such that $|x-4|, |y-6|, |z-12| < \frac{1}{\sqrt{3}}$

Ans:- $(0.6a - 0.8b)^2 + (0.8a + 0.6b)^2 = 0.36a^2 - \cancel{2\times0.6\times0.8ab} + 0.64b^2 + 0.64a^2 + \cancel{2\times0.6\times0.8ab} + 0.36b^2$

$= a^2 + b^2$

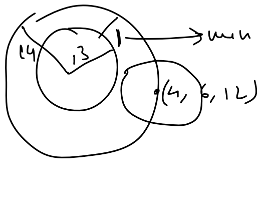
if the third number is said to be c

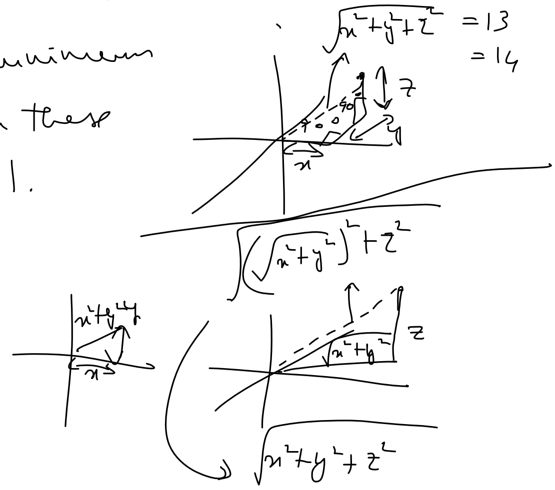
So we get, $a^2+b^2+c^2$ remains invariant which is $3^2+4^2+12^2$
But $4^2+6^2+12^2 \neq 3^2+4^2+12^2$ so not reachable

$a^2+b^2+c^2 = 3^2+4^2+12^2 = 13^2$ $\qquad$ $4^2+6^2+12^2 = 14^2$

$(x-4)^2 + (y-6)^2 + (z-12)^2 < 1$

(x, y, z) distance from $(4, 6, 12)$ is < 1. So (x, y, z) cannot reach $a^2+b^2+c^2 = 13^2$ as the minimum distance between these two spheres is 1.





HomeWork

Q) We have arranged 5 ones and 4 zeros around a circle in any order. Then between any two equal digits you write 0 and between different digits you write 1 and erase the original digits. Prove that we can never reach 9 zeros.